Algebra 2, Semester 1 2015
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Practice Final

Problem 1.

(1) What is the minimal polynomial of $\sqrt{2}+\sqrt{3}$ over $\mathbb{Q}$?
(2) What is the splitting field of the polynomial from part (1)?

Problem 2. Give examples of:

(a) An irreducible polynomial $f(x) \in \mathbb{Q}[x]$ of degree 5.
(b) A field extension of $\mathbb{F}_5$ of degree 3.
(c) A Galois field extension L/K such that $\mathrm{Gal}(L/K) = \mathbb{Z}/7$.
(d) A field extension of degree 3 that is not Galois.

Problem 3.

(a) State the definition of a solvable group.
(b) Show that the Dihedral group on 8 elements

$$G = \langle \sigma, \tau \mid \sigma^4 = \tau^2 = \mathrm{id}, \tau\sigma = \sigma^3\tau \rangle$$

is solvable.

Problem 4. Find the Galois groups of the splitting fields of the following polynomials over $\mathbb{Q}$.

(a) $x^4 - 2$
(b) $x^3 + 5x + 5$
(c) $x^4 + 3x + 3$

Hint: You may want to use the following formulae:

- the discriminant of $f(x) = x^3 + ax + b$ is $-4a^3 - 27b^2$.
- the discriminant of $f(x) = x^4 + ax + b$ is $-27a^4 + 256b^3$
- the resolvent cubic of $f(x) = x^4 + ax + b$ is $x^3 - 4bx - a^2$.

Problem 5.

(a) Express $\alpha_1^3+\alpha_2^3+\alpha_3^3$ as a polynomial in the elementary symmetric functions s_1, s_2, s_3 where

$$s_1 = \alpha_1 + \alpha_2 + \alpha_3$$
$$s_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3$$
$$s_3 = \alpha_1\alpha_2\alpha_3$$

(b) Let $\alpha_1, \alpha_2, \alpha_3$ be the roots of $f(x) = x^3 - x^2 + 2x - 3$. What is $\alpha_1^3 + \alpha_2^3 + \alpha_3^3$?

Problem 6.

(a) Prove that any degree 2 field extension $K \subseteq L$ is normal.
(b) Prove that if $f(x) \in \mathbb{Q}[x]$ is an irreducible polynomial with roots $\alpha, \beta \in \mathbb{C}$, then the fields $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\beta)$ are isomorphic.